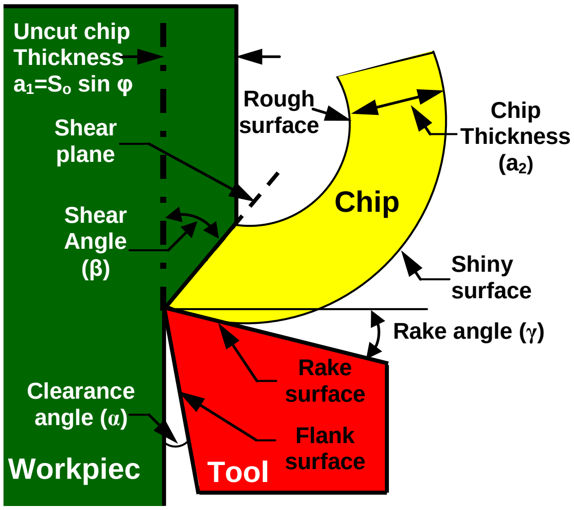
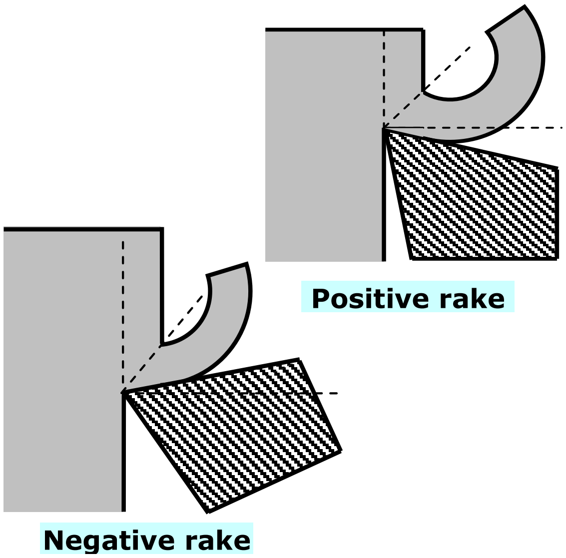
Chip Formation in Metal Machining

Since the practical machining is complex we use orthogonal cutting model to explain the mechanics.In this model we used wedge shaped tool. As the tool forced into the material the chip is formed by shear deformation.





Deformation of Uncut Layer

The problem in the study of the mechanism of chip formation is the deformation process of the chip ahead of the cutting tool. It is difficult to apply equation of plasticity as the deformations in metal cutting are very large. Experimental techniques have always been resorted to for analyzing the deformation process of chips. Several methods have been used:

- Taking photographs of the side surface of the chip with a high speed movie camera fitted with microscope.

- Observing the **grid deformation** (directly)

  - on the side surface of the work piece and
  - on the inner surface of a compound work piece.

- Examination of frozen chip samples taken by

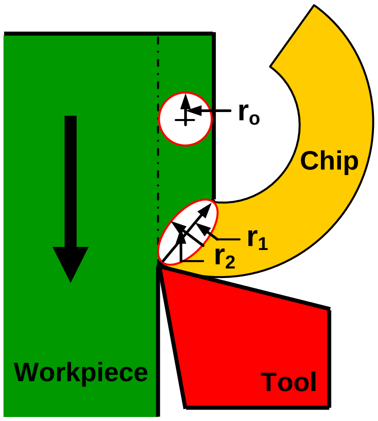
  - drop tool apparatus and
  - quick stop apparatus,

Grid Deformation Methods

The type of stress-state conditions is evaluated by means of an **angle index e** obtainable from **Levy-Lode's theorem**,

$$\frac{e_1+e_2-2e_3}{e_1-e_2}=\frac{\tan(30^o-e)}{\tan 30^o}\text{------[1]}$$

$$e_1=\ln\left(\frac{r_1}{r_o}\right),\ e_2=\ln\left(\frac{r_2}{r_o}\right)\quad \text{and } e_1+e_2+e_3=o\text{----[2]}$$



Schematic representation of the translocation of circles into ellipses during chip formation.

where,
e = deformation criteria
=0^0 for pure tension
=30^0 for pure shear
=60^0 for pure compression
r_o= radius of circles marked on the workpiece
r_1 & r_2= semi-axes of the ellipse after deformation.

From Equation [1] and Equation [2]

$$\frac{\tan(30^o-e)}{\tan 30^o}=\ln\left(\frac{r_1r_2}{r_o^{\ 2}}\right)^3\Big/\ln\left(\frac{r_1}{r_2}\right)\text{------[3]}$$

<u>Case-1: For Pure Tension [e=0]</u>

$$r_1=r_o(1+\varepsilon)\text{ and }r_2=r_o(1-\mu\varepsilon)\text{----------[4]}$$

$$\frac{r_1}{r_o}=1+\varepsilon,\ \frac{r_2}{r_0}=1-\frac{\varepsilon}{2}\ \text{ and }\left(\frac{r_2}{r_0}\right)^2=(1-2.\frac{\varepsilon}{2}-\frac{\varepsilon^2}{4})\cong(1-\varepsilon)\text{----[5]}$$

Where,
ε = cutting strength
μ = frictional coefficient=½
since **ε** is very very small so neglecting $\boldsymbol{\varepsilon}^2$

Now, from equation [5]

$$\left(\frac{r_1}{r_0}\right)\left(\frac{r_2}{r_0}\right)^2=\frac{r_1r_2^{\ 2}}{r_0^{\ 2}}=(1+\varepsilon)(1-\varepsilon)\cong 1\text{-----[6]}$$

From Equation [3] and Equation [6]

$$\frac{\tan(30^0-e)}{\tan 30^0}=\frac{\ln(\frac{r_1 r_2}{r_0^2})^3}{\ln(\frac{r_1}{r_2})}=\frac{\ln\left(\frac{r_1^{\;2} r_2^{\;4}}{r_0^{\;6}}\cdot\frac{r_1}{r_2}\right)}{\ln\left(\frac{r_1}{r_2}\right)}=1\text{----}[7]$$

$$\text{or, } \tan(30^0-e)=\tan 30^0$$

$$\text{or, } e=0^o \text{ for Pure Tension}$$

Case-2: For Pure Shear [e=30^0]

$$r_1=r_o(1+\varepsilon+\mu\varepsilon) \text{ and } r_2=r_o(1\text{-}\varepsilon\text{-}\mu\varepsilon)\text{----------}[8]$$

$$\frac{r_1}{r_o}=1+\frac{3}{2}\varepsilon,\ \frac{r_2}{r_0}=1-\frac{3}{2}\varepsilon \ \text{ and } \left(\frac{r_1}{r_0}\right)\left(\frac{r_2}{r_0}\right)=(1+\frac{3}{2}\varepsilon)\,(1+\frac{3}{2}\varepsilon)\ \cong 1\text{----}[9]$$

From Equation [3] and Equation [9]

$$\frac{\tan(30^0-e)}{\tan 30^0}=\frac{\ln\left(\frac{r_1 r_2}{r_0^{\;2}}\right)^3}{\ln\left(\frac{r_1}{r_2}\right)}=\frac{\ln(1)^3}{\ln\left(\frac{r_1}{r_2}\right)}=0\text{-----}[10]$$

$$\text{or, } \tan(30^0-e)=0=\tan(0)$$

$$\text{or, } e=30^o \text{ for Pure Shear}$$

Case-3: For Pure Compression [e=60^0]

$$r_1=r_o(1+\mu\varepsilon) \text{ and } r_2=r_o(1-\varepsilon)\text{----------}[11]$$

$$\frac{r_1}{r_o}=1+\frac{\varepsilon}{2},\ \frac{r_2}{r_0}=1-\varepsilon \ \text{ and } \left(\frac{r_1}{r_0}\right)^2=(1+2.\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4})\cong(1+\varepsilon)\text{ ----}[12]$$

$$\left(\frac{r_1}{r_o}\right)^2\left(\frac{r_2}{r_o}\right)=(1+\varepsilon)(1\text{-}\varepsilon)\cong 1\text{---------}[13]$$

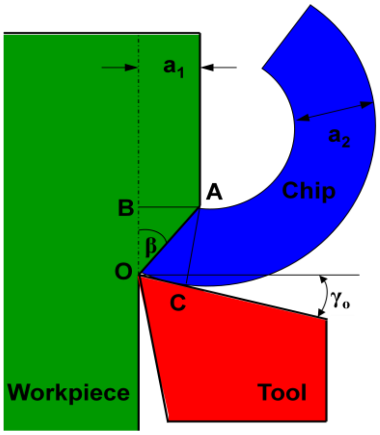
From Equation [3] and Equation [13]

$$\frac{\tan(30^0 - e)}{\tan 30^0} = \frac{\ln\left(\frac{r_1{}^2 r_2}{r_0{}^3}\right)^2 \left(\frac{r_2}{r_1}\right)}{\ln\left(\frac{r_1}{r_2}\right)} = -1 \text{ or, } \tan(30^0 - e) = -\tan 30^0 = \tan(-30^0)$$

$$\text{or, } e = 60^\circ \text{ for Pure Compression}$$

Chip Reduction Coefficient (ξ)

Chip reduction coefficient (ξ) is defined as the ratio of chip thickness (a2) to the uncut chip thickness (a1). This factor, ξ, is an index of the degree of deformation involved in chip formation process during which the thickness of layer increases and the length shrinks. In the USA, the inverse of ξ is denoted by rc and is known as cutting ratio. The following **Figure** shows the formation of flat chips under orthogonal cutting conditions. From the geometry of the following **Figure**.



$$\xi = \frac{a_2}{a_1} = \frac{AC}{AB} = \frac{OA\cos(\beta - \gamma_0)}{OA\sin\beta} = \frac{\cos\beta\cos\gamma_0 + \sin\beta\sin\gamma_0}{\sin\beta} ----[1]$$

Shear Angle (β)

From Equation [1]

$$\xi = \frac{\cos\beta\cos\gamma_0 + \sin\beta\sin\gamma_0}{\sin\beta} = \frac{\cos\gamma_0}{\tan\beta} + \sin\gamma_0$$

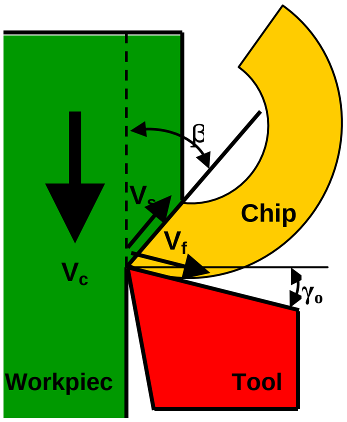
$$\tan\beta = \frac{\cos\gamma_0}{\xi - \sin\gamma_o}$$

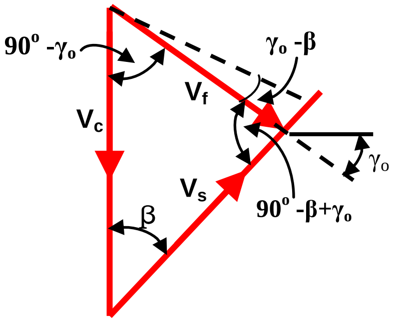
$$\beta = \tan^{-1}\left(\frac{\cos\gamma_o}{\xi - \sin\gamma_o}\right) \text{Shear angle}$$

Velocity Relationships

The following Figure shows the velocity relation in metal cutting. As the tool advances, the metal gets cut and chip is formed. The chip glides over the rake surface of the tool. With the advancement of the tool, the shear plane also moves. There are three velocities of interest in the cutting process which include:

V_C = velocity of the tool relative to the workpiece. It is called **cutting velocity**
V_f = velocity of the chip (over the tool rake) relative to the tool. It is called **chip flow velocity**
V_s= velocity of displacement of formation of the newly cut chip elements, relative to the workpiece along the shear plane. It is called **velocity of shear**





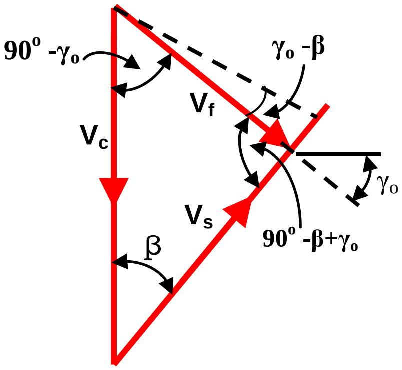
According to principles of kinematics, these three velocities, i.e. their vectors must form a closed velocity diagram. The vector sum of the cutting velocity, Vc, and the shear velocity, Vs, is equal to chip velocity, Vf. Thus,

$$\overline{V}_f=\overline{V}_c+\overline{V}_s$$

$$\frac{V_s}{\sin(90^{o}-\gamma_o)}=\frac{V_c}{\sin\left[90^{o}-(\beta-\gamma_o\right]}=\frac{V_f}{\sin\beta}$$

$$V_f=V_c\frac{\sin\beta}{\sin\left[90^{0}-(\beta-\gamma_o)\right]}=\frac{V_c\sin\beta}{\cos(\beta-\gamma_o)}=\frac{V_c}{\xi}$$

$$\text{or, }\frac{V_c}{V_f}=\xi$$



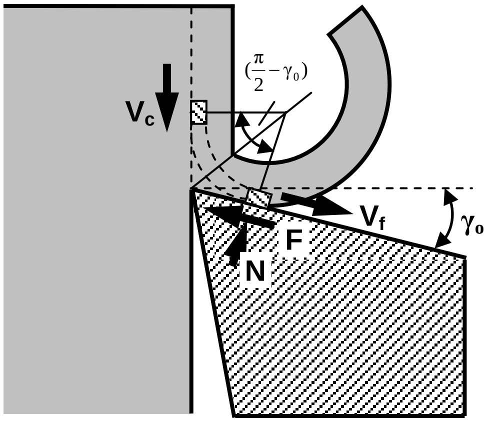
Kronenberg derived an interesting relation for chip reduction coefficient (ξ) which is of considerable physical significance. Considering the motion of any chip particle as shown in the following **Figure** to which principles of momentum change are applied:

$$F = -m\frac{dv}{dt}$$

$$N = m\omega^2 r = mv\frac{d\theta}{dt}$$

$$\mu = \frac{F}{N} = -\frac{dv}{v\,d\theta}$$

$$-\frac{dv}{v} = \mu\,d\theta$$

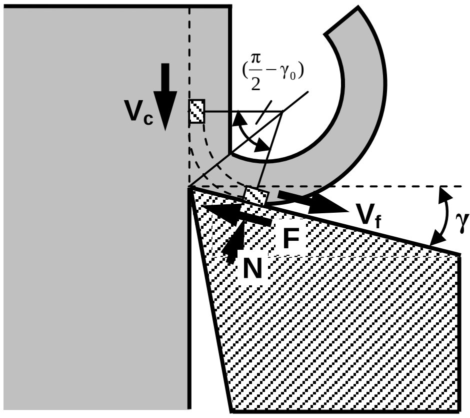


As the velocity changes from V_c to V_f, hence

$$\int_{V_c}^{V_f} -\frac{dv}{v} = \int_0^{(\frac{\pi}{2}-\gamma_o)} \pi d\theta$$

$$-\ln\left(\frac{V_f}{V_c}\right) = \mu\left(\frac{\pi}{2}-\gamma_o\right)$$

$$\frac{V_c}{V_f} = e^{\mu\left(\frac{\pi}{2}-\gamma_0\right)}$$



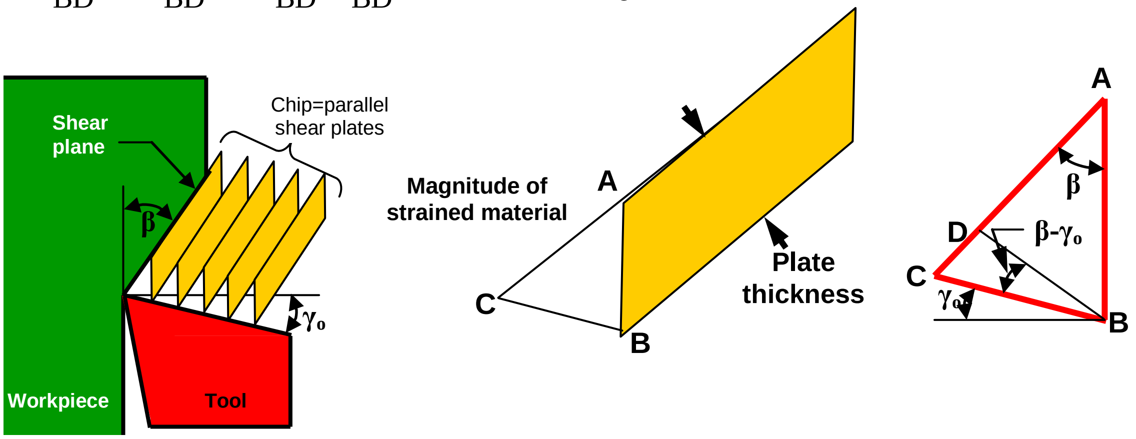
$$\xi = e^{\mu\left(\frac{\pi}{2}-\gamma_0\right)}$$

This equation demonstrates that the chip reduction coefficient and chip flow velocity is dependant on the frictional aspects at the interface as well as the orthogonal rake angle (γ_0). If γ_0 is increased, chip reduction coefficient decreases.

Shear Strain (ε)

The value of the **shear strain** (ε) is an indication of the amount of deformation that the metal undergoes during the process of chip formation. The shear strain that occurs along the shear plane can be estimated by examining the following Figure. The shear strain can be expressed as follows:

$$\varepsilon = \frac{AC}{BD} = \frac{AD + CD}{BD} = \frac{AD}{BD} + \frac{CD}{BD} = \cot\beta + \tan(\beta - \gamma_o) \text{ - -}[1]$$



Shear strain during chip formation (a) chip formation depicted as a series of parallel sliding relative to each other (b) one of the plates isolated to illustrate the definition of shear strain based on this parallel plate model (c) shear strain triangle

From equation [1]

$$\varepsilon = \cot\beta + \tan(\beta - \gamma_o) = \frac{\cos\gamma_o}{\sin\beta.\cos(\beta - \gamma_o)} \text{ --}[2]$$

From velocity relationship

$$\frac{V_s}{V_c} = \frac{\cos\gamma_o}{\cos(\beta - \gamma_o)} \text{ ---}[3]$$

From equation [2] and equation [3]

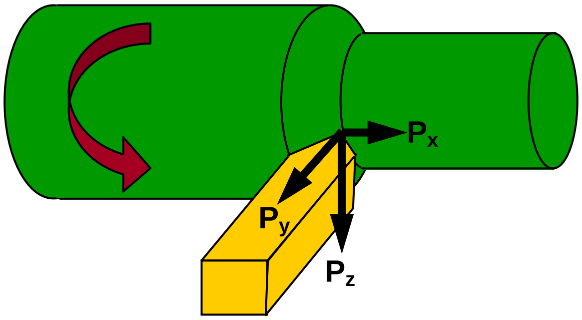
$$\varepsilon = \frac{V_s}{V_c \sin\beta} \text{ Shear strain}$$

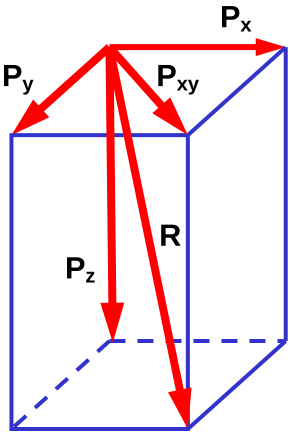
Mechanics of Metal Cutting

The force acting on a cutting tool during the process of metal cutting are the fundamental importance in the design of cutting tools. The determination of cutting forces necessary for deformation the work material at the shear zone is essential for several important requirements:

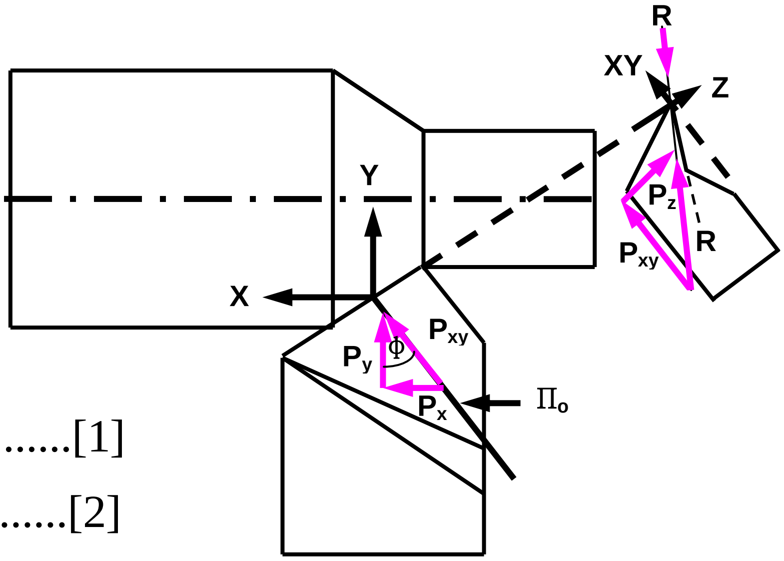
- to estimate the power requirements of a machine tool
- to estimate the straining actions that must be resisted by the machine tool components, bearings, jigs and fixtures
- to evaluate the role of various parameters in cutting forces
- to evaluate the performance of any new work material, tool material, environment, techniques etc. with respect to machinability (cutting forces)

The force system in the general case of conventional turning process is shown in the following **Figure.**





Px = **feed force** in the direction of the tool travel
Py = **thrust force** in the direction perpendicular to the produced surface
Pz = **cutting force** or **main force** acting in the direction of the cutting velocity.



$$P_x = P_{xy}\sin\varphi..................[1]$$

$$P_y = P_{xy}\cos\varphi.................[2]$$

Several forces can be defined relative to the orthogonal cutting model. Based on these forces, shear stress, coefficient of friction, and certain other relationships can be defined.

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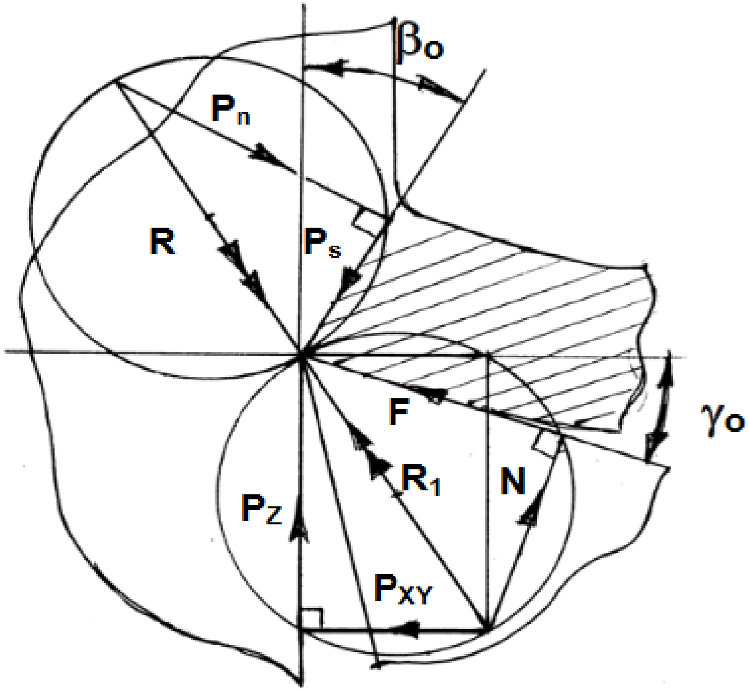
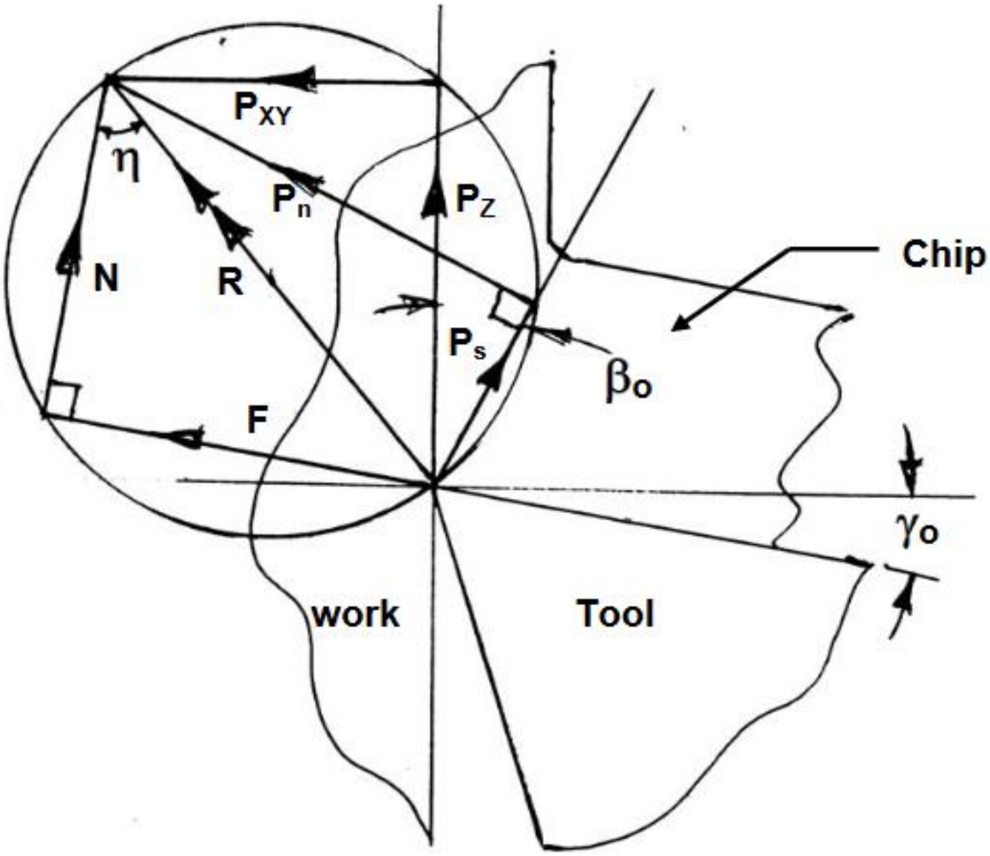


. 8.5 *Development of Merchants Circle Diagram.*

Merchant's Circle Diagram with cutting forces



Advantageous use of Merchant's Circle Diagram

- Easy, quick and reasonably accurate determination of several other forces from a few known forces involved in machining
- Friction at chip-tool interface and dynamic yield shear strength can be easily determined
- Equations relating the different forces are easily developed.

Some limitations of use of MCD

- Merchant's Circle Diagram(MCD) is valid only for orthogonal cutting
- by the ratio, F/N, the MCD give s apparent (not actual) coefficient of friction
- It is based on single shear plane theory.

Merchant Circle Diagram (MCD)

The following relationships suggest a circle representation of forces as done by **Merchant** and indicated in the following **Figure**.

$$\overline{R} = \overline{F} + \overline{N} = \overline{P}_s + \overline{P}_n = \overline{P}_z + \overline{P}_{xy} \text{.......[3]}$$

$$F = P_z \sin\gamma_o + P_{xy}\cos\gamma_o \text{.....................[4]}$$

$$N = P_z \cos\gamma_o - P_{xy}\sin\gamma_o \text{....................[5]}$$

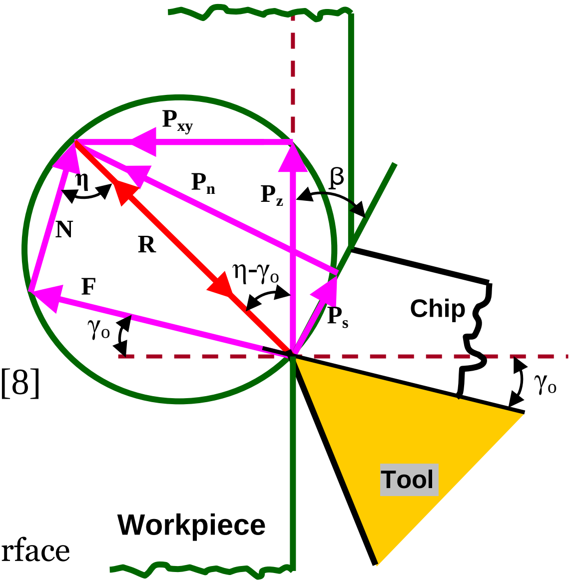
$$P_s = P_z \cos\beta - P_{xy}\sin\beta \text{......................[6]}$$

$$P_n = P_z \sin\beta + P_{xy}\cos\beta \text{.....................[7]}$$

From Equation [4] and [5]

$$\mu = \frac{F}{N} = \frac{P_z \sin\gamma_o + P_{xy}\cos\gamma_o}{P_z \cos\gamma_o - P_{xy}\sin\gamma_o} = \tan\eta \text{.....[8]}$$

Where,
μ = kinetic coefficient of friction
η = mean angle of friction at the rake surface



From the geometry of force relations of MCD circle

$$P_z = R\cos(\eta - \gamma_0) \text{................[9]}$$

$$P_s = R\cos(\beta + \eta - \gamma_0) \text{...........[10]}$$

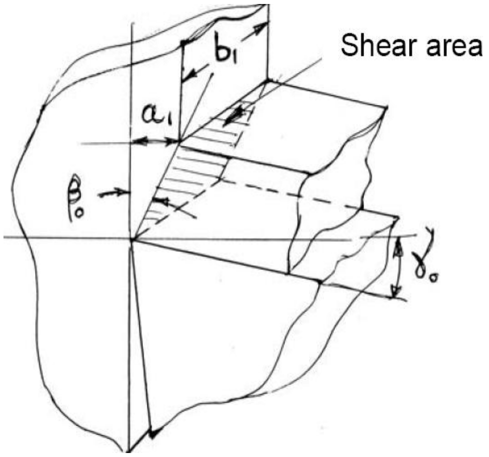
Fron Equation [9] and [10]

$$P_z = P_s\left[\frac{\cos(\eta - \gamma_0)}{\cos(\beta + \eta - \gamma_o)}\right] \text{..........[11]}$$

Based on the shear force, the shear stress (τ_s) which acts along the shear plane between the work and the chip is:

$$\tau_s = \frac{P_s}{A_s}, \text{where As} = \text{area of the shear plane} = \frac{S_o\, t}{\sin\beta}$$

$$\tau_s = \frac{P_s \sin\beta}{S_o\, t} \text{..........[12]}$$



From Equation [11]and [12]

$$P_z = \tau_s\, S_0\, t\left[\frac{\cos(\eta-\gamma_0)}{\sin\beta\, \cos(\beta+\eta-\gamma_o)}\right]...........[13]$$

Similarly,

$$P_{xy} = \tau_s\, S_0\, t\left[\frac{\sin(\eta-\gamma_0)}{\sin\beta\, \cos(\beta+\eta-\gamma_o)}\right].......[14]$$

In metal cutting one of the main problem is to evaluate the cutting forces P_z and P_{xy} from the given cutting conditions and initial properties of work material and it is necessary to determine τ_s, β and η by suitable relationships.

Earnest-Merchant Theory

Earnest and Merchant extended their analysis and studied the relationship between the shear angle and the cutting conditions. They suggested that the shear angle always takes the value that reduces the total energy consumed in cutting to a minimum. Because the total work done in cutting is dependent upon and is a direct function of the component P_z of the cutting force, they developed an expression for P_z in terms of β and the constant properties of the workpiece material. Condition for maximum cutting force (P_z) from Equation [13]

$$\frac{dP_z}{d\beta} = 0, \text{or}, \frac{dP_z}{d\beta} = \frac{d}{d\beta}\left[\frac{\tau_s S_0 t}{\sin\beta} \cdot \frac{\cos(\eta-\gamma_0)}{\cos(\beta+\eta-\gamma_o)}\right] = 0$$

$$\tau_s S_0\, t\cos(\eta-\gamma_0)\left[\frac{\cos\beta\cos(\beta+\eta-\gamma_0)-\sin\beta\sin(\beta+\eta-\gamma_0)}{\{\sin\beta\cos(\beta+\eta-\gamma_0\}^2}\right] = 0$$

$$\cos\beta\cos(\beta+\eta-\gamma_0)-\sin\beta\sin(\beta+\eta-\gamma_0) = 0, \text{or } \cos(\beta+\beta+\eta-\gamma_0) = 0 = \cos\left(\frac{\pi}{2}\right)$$

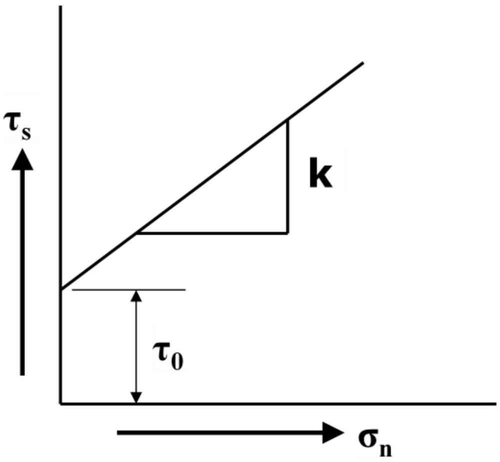
$$\beta = \frac{\pi}{4} - \frac{\eta}{2} + \frac{\gamma_0}{2}..................[15]$$

Combining Equation [13] and [15]

$$P_z = 2\tau_s S_0\, t\cot\beta.............[16]$$

Merchant modified the relationship derived by Earnest-Merchant, by assuming that the shear stress along the shear plane varies linearly with normal stress (σ_n). It is given as (from the following Figure).

$\tau_s = \tau_0 + k\,\sigma_n$................[17]



From the geometry of force relations of MCD

$P_s = R\cos(\beta+\eta-\gamma_0)$ and $P_n = R\sin(\beta+\eta-\gamma_0)$................[18]

$$P_n = P_s\tan(\beta+\eta-\gamma_0)$$

$$\frac{P_n}{A_s} = \frac{P_s}{A_s}\tan(\beta+\eta-\gamma_0)$$

$\sigma_n = \tau_s\tan(\beta+\eta-\gamma_0)$....................[19]

From Eqation [17] and [19]

$\tau_s = \left[\dfrac{\tau_0}{1-k\tan(\beta+\eta-\gamma_0)}\right]$.......................[20]

Combining Equation [13] and [20]

$P_z = \dfrac{\tau_o S_0 t\cos(\eta-\gamma_0)}{\sin\beta\cos(\beta+\eta-\gamma_0)\{1-k\tan(\beta+\eta-\gamma_0)\}}$...............[21]

Condition for maximum cutting force (Pz) from Equation [21]

$$\frac{dP_z}{d\beta} = 0, \text{ or } \frac{dP_z}{d\beta} = \frac{d}{d\beta}\frac{\tau_o S_0 t\cos(\eta-\gamma_0)}{\sin\beta\cos(\beta+\eta-\gamma_0)\{1-k\tan(\beta+\eta-\gamma_0)\}} = 0$$

$$or\ \frac{\tau_s S_0 t\cos(\eta-\gamma_0)\begin{bmatrix}\cos\beta\cos(\beta+\eta-\gamma_0)-\sin\beta\sin(\beta+\eta-\gamma_0)- \\ k\cos\beta\sin(\beta+\eta-\gamma_0)-k\sin\beta\cos(\beta+\eta-\gamma_0)\end{bmatrix}}{[\sin\beta\cos(\beta+\eta-\gamma_0)-k\sin\beta\sin(\beta+\eta-\gamma_0)]^2} = 0$$

$$\cos(2\beta+\eta-\gamma_0)-k\sin(2\beta+\eta-\gamma_0) = 0$$

$$\cot(2\beta+\eta-\gamma_0) = k$$

$2\beta+\eta-\gamma_0 = \cot^{-1}(k) = c \cong 80^0 \text{ to } 85^0$...........................[22]

From Equation [21] and [22]

$P_z = \tau_s S_0 t[\cot\beta+\tan(c-\beta)]$..........[23]

Lee and Shaffer Theory

According to this theory the shear occurs on a single plane. So for a cutting process according to this theory, the following are supposed to hold good:

- The material ahead of the cutting tool behaved as ideal plastic material
- The chip does not get hardened
- The chip and parent work material are separated by a shear plane.

Lee and Shaffer derived the following relationship as:

$$\beta + \eta - \gamma_0 = \frac{\pi}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots [24]$$

From Equation [13] and [24]

$$P_z = \tau_s \, S_0 \, t\left(\cot\beta + 1\right) \ldots\ldots\ldots\ldots [25]$$

Where,

$$\cot\beta = \frac{1}{\tan\beta} = \frac{\xi - \sin\gamma_o}{\cos\gamma_o} \cong \xi - \tan\gamma_o$$

$$P_z = \tau_s \, S_0 \, t\left(\xi - \tan\gamma_0 + 1\right) \ldots\ldots\ldots\ldots [25]$$

Sample Mathematical problems:

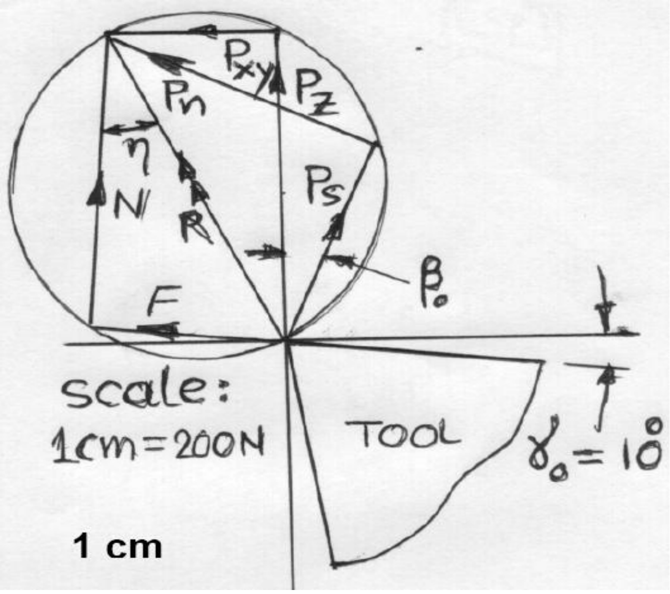
During turning a ductile alloy by a tool of $\gamma_O = 10^O$, it was found P_Z= 1000 N, P_X= 400 N, P_Y= 300 N and $\zeta = 2.5$. Evaluate, using MCD, the values of F, N and μ as well as P_s and P_n for the above machining.

Solution:

- force, $P_{XY} = \sqrt{P_X^2 + P_Y^2}$ = $\sqrt{(400)^2 + (300)^2}$ = 500 N
- Select a scale: 1 cm=200N
- Draw the tool tip with $\gamma_o = 10^O$
  In scale, P_Z=1000/200= 5 cm and P_{XY}=500/200=2.5cm

- Draw P_Z and P_{XY} in the diagram
- Draw R and then the MCD
- Extend the rake surface and have F and N as shown
- Determine shear angle, β_O

  $\tan \beta_o = \cos\gamma_o / (\zeta - \sin\gamma_o)$
  $= \cos10^O/(2.5 - \sin10^O) = 0.42$
  $\beta_o = \tan^{-1}(0.42) = 23^O$
- Draw P_S and P_n in the MCD
- From the MCD, find F = 3x200 = 600 N; N = 4.6x200 = 920 N;
  μ = F/N = 600/920 = 0.67
  P_S = 3.4x200 = 680; P_n= 4.3x200 = 860 N



<u>Above problem can be solved by using only equations.(without MCD)</u>

Q. 3 During turning a steel rod of diameter 150 mm by a carbide tool of geometry;

0^o, $—12^o$, 8^o, 6^o, 15^o, 60^o, 0 (mm)

at speed 560 rpm, feed 0.32 mm/rev. and depth of cut 4.0 mm the followings were observed :

P_Z= 1000 N, P_Y= 200 N, a_2=0.8 mm

Determine, without using MCD, the expected values of F, N, μ, P_S, P_n, τ_s, cutting power and specific energy requirement for the above mentioned machining operation.

Solution :

- $P_{XY} = P_X/\sin\phi = 200/\cos 60^o = 400$ N
- $F = P_Z \sin\gamma_o + P_{XY}\cos\gamma_o$;

Here $\gamma_o = - 12^o \backslash \sin\gamma_o = - 0.208$ and $\cos\gamma_o = 0.978$

$F = 1000(-0.208) + 400(0.978) = 600$ N ans.

and $N = P_Z\cos\gamma_o — P_{XY}\sin\gamma_o$

$= 1000(0.978) — 400(-0.208)$

$= 1060$ N answer

So, $\mu_a = F/N = 600/1060 = 0.566$ answer

- $P_S = P_Z\cos\beta_o — P_{XY}\cos\beta_o$

where $\beta_o = \tan^{-1}(\cos\gamma_o/(\zeta—\sin\gamma_o))$

Here, $\zeta = a_2/(s_o \sin\phi) = 0.8/(0.32\text{x}\sin 60^o) = 2.88$

$\beta_o= \tan^{-1}\{(0.978/(2.88+0.208)\} = 17.6^o$

So, P_S= 1000xcos(17.6^o) – 400xsin(17.6^o) = 832 N **answer**

and P_N= 1000sin(17.6^o) + 400cos(17.6^o) = 683 N **answer**

- $P_S = (ts_o\ \tau_s)/\sin\gamma_o$

$\therefore \tau_s = P_S \sin\gamma_o /(ts_o)$ = 832sin(17.6^o)/(4x0.32)

= 200 N/mm^2 answer

- Cutting power, $P_C = P_Z.V_C$

where $V_C = \pi DN/1000 = \pi\text{x}150\text{x}560/1000 = 263$ m/min

$\therefore P_C$ = 1000x263 N-m/min = 4.33 KW **answer**

- Specific energy consumption, E_C

E_C= power/MRR = $(P_Z.V_C)/(V_C.s_o.t)$ N-m/m-mm^2

= 1000x263 (Joules/min)/{263x0.32x4x1000(mm^3/min)}

= 0.78 Joules/mm^3 **answer**

In a machining operation that approximates orthogonal cutting, the cutting tool has a rake angle = 10°. The chip thickness before the cut $t_o = 0.50$ mm and the chip thickness after the cut $t_c = 1.125$ in. Calculate the shear plane angle and the shear strain in the operation.

Solution: The chip thickness ratio can be determined from Eq. (21.2):

$$r = \frac{0.50}{1.125} = 0.444$$

The shear plane angle is given by Eq. (21.3):

$$\tan \phi = \frac{0.444 \cos 10}{1 - 0.444 \sin 10} = 0.4738$$

$$\phi = 25.4^\circ$$

Finally, the shear strain is calculated from Eq. (21.4):

$$\gamma = \tan(25.4 - 10) + \cot 25.4$$

$$\gamma = 0.275 + 2.111 = 2.386$$

Necessary equations:

$$\beta = \tan^{-1}\left(\frac{\cos\gamma_o}{\xi - \sin\gamma_o}\right) \text{Shear angle}$$

$$\frac{V_c}{V_f} = \xi$$

$$\frac{V_s}{\sin(90^o - \gamma_o)} = \frac{V_f}{\sin\beta}$$

$$\xi = e^{\mu\left(\frac{\pi}{2} - \gamma_0\right)}$$

$$\varepsilon = \cot\beta + \tan(\beta - \gamma_o)$$

$$\varepsilon = \frac{V_s}{V_c \sin\beta} \text{Shear strain}$$

$$P_x = P_{xy}\sin\varphi$$

$$P_y = P_{xy}\cos\varphi$$

$$\overline{R} = \overline{F} + \overline{N} = \overline{P}_s + \overline{P}_n = \overline{P}_z + \overline{P}_{xy}$$

$$F = P_z \sin\gamma_o + P_{xy}\cos\gamma_o$$

$$N = P_z \cos\gamma_o - P_{xy}\sin\gamma_o$$

$$P_s = P_z \cos\beta - P_{xy}\sin\beta$$

$$P_n = P_z \sin\beta + P_{xy}\cos\beta$$

$$\mu = \frac{F}{N} = \frac{P_z \sin\gamma_o + P_{xy}\cos\gamma_o}{P_z \cos\gamma_o - P_{xy}\sin\gamma_o} = \tan\eta$$

$$P_z = R\cos(\eta - \gamma_0)$$

$$P_s = R\cos(\beta + \eta - \gamma_0)$$

$$P_z = P_s\left[\frac{\cos(\eta - \gamma_0)}{\cos(\beta + \eta - \gamma_o)}\right]$$

$$P_z = \tau_s S_0 t\left[\frac{\cos(\eta - \gamma_0)}{\sin\beta\cos(\beta + \eta - \gamma_o)}\right]$$

$$P_{xy} = \tau_s S_0 t\left[\frac{\sin(\eta - \gamma_0)}{\sin\beta\cos(\beta + \eta - \gamma_o)}\right]$$

$$\boldsymbol{\beta = \frac{\pi}{4} - \frac{\eta}{2} + \frac{\gamma_0}{2}}$$

$$P_z = 2\tau_s S_0 t\cot\beta \tau_s = \left[\frac{\tau_0}{1 - k\tan(\beta + \eta - \gamma_0)}\right]$$

$$P_z = \tau_s S_0 t\left[\cot\beta + \tan(c - \beta)\right]$$

$$\beta + \eta - \gamma_0 = \frac{\pi}{4}$$

$$P_z = \tau_s S_0 t\left(\xi - \tan\gamma_0 + 1\right)$$